

PLANE FLOWS WITH CLOSED ORBITS⁽¹⁾

BY
ANATOLE BECK

1. **Introduction.** One-parameter flows in topological and measure spaces, and especially in 2- and 3-dimensional Euclidean spaces, occur often in the consideration of various mathematical structures. Much of differential equation theory is related to this concept, not to mention such literal applications as aero- or hydro-dynamics. The flow represents the points of the space as idealized particles moving under a group of transformations indexed by the real line (time). The transformations could be, for example, measure-preserving and continuous (as in incompressible viscous flow), or continuously differentiable (as in the solutions to certain classes of differential equations), or merely measure-preserving (as in stochastic processes). Going further, it is interesting to know what can be said of such flows under particular sets of hypotheses. In this paper, we will be concerned with continuous flows in the Euclidean plane.

For each flow, we consider the set of points which are fixed under every transformation. This set is an invariant of the flow; flows which are homeomorphically equivalent have homeomorphic invariant sets. In a previous paper [1], we show that every closed set in the plane can be represented as the fixed point set of some continuous flow. The solution, as it is constructed there, has many orbits (or trajectories) slowing down and approaching the fixed point set asymptotically in time. The points to which they converge, which we may call *stagnation points* after the aerodynamic usage, proliferate in this example in a way which is not observed in, say, fluid dynamics. In fact, the prevalence of stagnation points creates the feeling that the question answered is the wrong question, in a certain important sense. It is thus natural to inquire into the behavior of flows in which there are few or no stagnation points. This could be guaranteed by the stronger assumption that every orbit is closed, and this is the condition we impose on the flows in this paper. We shall see that not all closed sets can be invariant sets of such flows and, in fact, we show that a countable set of points cannot be an invariant set if it has a limit point. In this paper, we shall give a topological characterization of the sets which are invariant sets of flows with closed orbits (the conditions are necessary and sufficient). In connection with this undertaking, we ask and answer the same questions

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for flows in which the orbits are compact and, in fact, the solution of this latter case is the major tool in examining the former.

2. Definitions. Let X be a topological space, and let $\phi(t, x)$ be a continuous mapping from $R \times X$ (where R is the real line) onto X for which

$$\phi(t_1, \phi(t_2, x)) = \phi(t_1 + t_2, x).$$

Then ϕ is called a *continuous one-dimensional flow in X* , or merely a *flow in X* . For each $x \in X$, let $\mathcal{O}(x) = \{\phi(t, x) | t \in R\}$ be called the *orbit of x (under ϕ)*. Then it is clear that $\mathcal{O}(x)$ is a simple open arc, a simple closed curve, or a single point, according as there are no real numbers $t > 0$ for which $\phi(t, x) = x$, or a least such $t > 0$, or arbitrarily small $t > 0$ satisfying this condition. For the purposes of this paper, a simple arc will be called a *line*, and a simple closed curve a *circle*. We shall use the words *straight line* and *genuine circle* when we mean these. A point whose orbit (under ϕ) is a single point will be called an *invariant point (of ϕ)*. The set F of all invariant points is called *the invariant set (of ϕ)*. It is clear that for each fixed $t \in R$, the mapping $x \rightarrow \phi(t, x)$ is a homeomorphism in X , and that $x = \phi(0, x)$ for all $x \in X$. We define a function $p(x)$, $x \in X$, by

$$\begin{aligned} p(x) &= \inf\{t > 0 | \phi(t, x) = x\} \\ &= \infty \text{ if } \phi(t, x) \neq x, \text{ all } t > 0. \end{aligned}$$

3. LEMMA. p is lower semi-continuous.

Proof. Let $\{x_\alpha | \alpha \in A\}$ be a net convergent to x_0 . If $p(x_0) = 0$, the result is trivially true. If $0 < p(x_0) \leq \infty$, then the denial of the assertion implies the existence of a $\bar{t} < p(x_0)$ and a subnet $\{x_{\alpha_\beta} | \beta \in B\}$ such that $p(x_{\alpha_\beta}) \leq \bar{t}$. For each $\beta \in B$, let t_β be chosen so that $\bar{t}/2 \leq t_\beta \leq \bar{t}$ and $\phi(t_\beta, x_{\alpha_\beta}) = x_{\alpha_\beta}$. Choose a subsubnet $\{x_{\alpha_{\beta_\gamma}} | \gamma \in \Gamma\}$ so that $\{t_{\beta_\gamma}\}$ converges, say to t_0 . Then

$$\begin{aligned} x_0 &= \lim x_{\alpha_{\beta_\gamma}} = \lim \phi(t_{\beta_\gamma}, x_{\alpha_{\beta_\gamma}}) = \phi(\lim t_{\beta_\gamma}, \lim x_{\alpha_{\beta_\gamma}}) \\ &= \phi(t_0, x_0) \neq x_0, \text{ since } 0 < t_0 < p(x_0), \end{aligned}$$

which is a contradiction. Thus, the lemma is proved. Q. E. D.

4. LEMMA. If $X = E^2$, and if the orbits under ϕ are all closed, then if $x_i \rightarrow x$ and $p(x) \neq 0$, we have $p(x_i) \rightarrow p(x)$ ⁽²⁾.

Proof. Working from Lemma 3, we need only prove that if $p(x_0) > 0$, $x_i \rightarrow x_0$, then $p(x_i)$ cannot be bounded away from $p(x_0)$ above. Thus, we need only consider the case $p(x_0) < \infty$, and we can take $p(x_0) = 1$. Assume, contrarily, that $x_i \rightarrow x_0$, $p(x_i) \geq 1 + \delta$, $\delta > 0$. Let M be the diameter of $\mathcal{O}(x_0)$, and let $t_0 > 0$ be chosen so that $t_0 < 1/2$, $t_0 < \delta/4$ and $\text{diam}(B_1) < M/4$, where $B_1 = \{\phi(t, x_0) | -t_0 \leq t \leq t_0\}$. Then $\text{diam}(B_0) > 3M/4$,

⁽²⁾ Actually, we do not use the fact that all the orbits are closed, only the orbits $\mathcal{O}(x_i)$. Thus, if $p(x_i)$ does not converge to $p(x)$, then $p(x_i) = \infty$ for infinitely many i .

where $B_0 = \{\phi(t, x_0) \mid t_0 < t < 1 - t_0\}$. Let ϵ_1 be chosen so that the ϵ_1 -neighborhoods of $\phi(-t_0, x_0)$, x_0 , $\phi(t_0, x_0)$ are pairwise disjoint, and so that $\epsilon_1 < M/32$. Let ϵ_2 be chosen so that

$$d(x_0, y) > \epsilon_2 \Rightarrow d(\phi(t, x_0), \phi(t, y)) < \epsilon_1, \text{ all } -2t_0 \leq t \leq 1 + 2t_0,$$

and also so that $d(x_0, B_0) > 2\epsilon_2$. Let ϵ be chosen so that $\epsilon < \epsilon_2$ and

$$d(x_0, y) < \epsilon \Rightarrow d(\phi(t, x_0), \phi(t, y)) < \epsilon_2, \text{ all } -t_0 \leq t \leq 1 + t_0.$$

We will show that any x_i such that $d(x_0, x_i) < \epsilon$ and $p(x_i) > 1 + \delta$ cannot have a closed orbit. Since $p(x_i) \geq 1 + \delta > 1 + 4t_0$, the arc

$$A = \{\phi(t, x_i) \mid -t_0 \leq t \leq 1 + t_0\} \text{ is simple.}$$

Since $2\epsilon_2 < d(x_0, B_0)$, no point within ϵ_2 of x_0 can lie within ϵ_2 of B_0 . Since every point of $A_0 = \{\phi(t, x_i) \mid t_0 \leq t \leq 1 - t_0\}$ lies within ϵ_2 of the corresponding point of B_0 , the intersection of A with the ϵ_2 -neighborhood of x_0 is contained in the arcs

$$A' = \{\phi(t, x_i) \mid -t_0 \leq t \leq t_0\} \quad \text{and} \quad A'' = \{\phi(t, x_i) \mid 1 - t_0 \leq t \leq 1 + t_0\}.$$

Let L be a line segment lying inside the ϵ_2 -neighborhood of x_0 having for endpoints $x' = \phi(t', x_i) \in A'$ and $x'' = \phi(t'', x_i) \in A''$, and including no other points of A . We now examine the three arcs C_0, C_1, C_2 described below.

$$C_0 = \{\phi(t, x_i) \mid t' + t_0 \leq t \leq t''\},$$

$$C_1 = L \cup \{\phi(t, x_i) \mid t' \leq t \leq t' + t_0\},$$

$$C_2 = \{\phi(t, x_i) \mid t'' \leq t \leq t'' + t_0\} \cup \phi(t_0, L).$$

We easily see that each arc joins $a = \phi(t' + t_0, x_i)$ to $b = \phi(t'', x_i)$, that each is simple, and that they are pairwise disjoint, since L lies in the ϵ_1 -neighborhood of x_0 , while $\phi(t_0, L)$ lies in the ϵ_1 -neighborhood of $\phi(t_0, x_0)$, which are disjoint. Further, C_0 lies within ϵ_1 of B_0 and C_1 and C_2 each lie within ϵ_1 of B_1 . Therefore $\text{diam}(C_0) > 3M/4 - 2\epsilon_1 > 3M/4 - 2M/32 = 11M/16$, while $\text{diam}(C_1) < M/4 + 2\epsilon_1 < M/4 + 2M/32 = 5M/16$, and similarly $\text{diam}(C_2) < 5M/16$. It thus follows, since $\text{diam}(C_0) > \text{diam}(C_1) + \text{diam}(C_2)$, that one of the arcs C_1, C_2 lies in the interior of the closed curve formed by the other and C_0 , except for its endpoints. Assume it is C_2 ; the other case is dual. Note that if

$$D_1 = C_0 \cup C_1 = \{\phi(t, x_i) \mid t' < t \leq t''\} \cup L,$$

while

$$D_2 = C_0 \cup C_2 = \{\phi(t, x_i) \mid t' + t_0 \leq t \leq t'' + t_0\} \cup \phi(t_0, L),$$

we have $\phi(t_0, D_1) = D_2$. Since $\phi(t_0, \cdot)$ is a homeomorphism, it takes $\text{Int}(D_1)$ onto $\text{Int}(D_2)$, so that

$$\text{Int}(D_1) \supset \text{Int}(\phi(t_0, D_1)) \supset \text{Int}(\phi(2t_0, D_1)) \supset \dots \supset \text{Int}(\phi(nt_0, D_1))$$

all $n > 0$. Also $\{\phi(t, x_i) | t'' < t \leq t'' + t_0\} \subset \text{Int}(D_1)$, so that

$$\{\phi(t, x_i) | t'' + nt_0 < t \leq t'' + (n+1)t_0\} \subset \text{Int}(\phi(nt_0, D_1)) \subset \text{Int}(D_1),$$

all $n > 0$, from which we have $\phi(t, x_i) \in \text{Int}(D_1)$ for all $t > t''$. Since $\phi(t', x_i) \notin \text{Int}(D_1)$, $\mathcal{O}(x_i)$ is not a circle, and thus not compact. Since $\{\phi(t, x_i) | t \geq t''\}$ is bounded but not compact, it cannot be closed. Since it is a closed subset of $\mathcal{O}(x_i)$, $\mathcal{O}(x_i)$ is not closed. Q. E. D.

5. THEOREM. *If ϕ is a continuous one-dimensional flow in the plane, and if all the orbits under ϕ are compact, then the set of points in the plane not fixed under the flow is the union of pairwise disjoint open annuli⁽³⁾.*

REMARK. By "open annulus" we mean not necessarily the open set lying between concentric circles (which we will call a *genuine open annulus*) but rather a homeomorph of such a set.

Proof. Assume $x \in E^2$ is not a fixed point. Then $\mathcal{O}(x)$ is a circle and there is a neighborhood of $\mathcal{O}(x)$ which contains no fixed point. Following the proof of Lemma 4, it is clear that for y sufficiently close to x , the period of y is arbitrarily close to that of x . Let z_0 be a point in the interior of $\mathcal{O}(x)$, and let the η -neighborhood of z_0 lie in this interior. Chose an ϵ_1 so small that $d(x, y) < \epsilon_1 \Rightarrow d(\phi(t, x), \phi(t, y)) < \eta/4$ for $0 \leq t \leq 2p(x)$. Then choose $\delta_1 > 0$ so that $d(x, \phi(t, x)) < \eta/4$ if $-\delta_1 < t < \delta_1$ and choose ϵ_2 so that $d(x, y) < \epsilon_2 \Rightarrow |p(x) - p(y)| < \delta_1$. Then for all y with $d(x, y) < \epsilon_x = \min(\epsilon_1, \epsilon_2)$, $\mathcal{O}(y)$ must have z_0 in its interior. Thus, if y_1 and y_2 are both within ϵ_x of x , $\mathcal{O}(y_1) = \mathcal{O}(y_2)$ or one of these contains the other in its interior.

Repeating the argument for each $z \in \mathcal{O}(x)$, we obtain an ϵ_z so that if y is within ϵ_z of z , $\mathcal{O}(y)$ contains z_0 in its interior. Using the compactness of $\mathcal{O}(x)$, we obtain an $\epsilon > 0$ such that if y lies within ϵ of $\mathcal{O}(x)$, $\mathcal{O}(y)$ is a circle with z_0 in its interior. Thus, if $U = \{y | y' \in \mathcal{O}(y) \Rightarrow d(y', \mathcal{O}(x)) < \epsilon\}$, then U is an annulus of orbits about $\mathcal{O}(x)$. We note that if U and V are such annuli, then so is $U \cup V$, and that if $\{U_i\}$ is a sequence of these with $U_i \subset U_{i+1}$, all $i > 0$, then $\bigcup_{i=1}^{\infty} U_i$ is also one. Let us choose any annulus U_1 of orbits about $\mathcal{O}(x)$. Then either all the boundary points of U_1 are fixed points or some are not. If $z \in \bar{U}_1 - U_1$ is not a fixed point, then there is an annulus V of orbits about $\mathcal{O}(z)$. Since U and V have a common point, they have a common orbit, and thus $U \cup V$ is an annulus of orbits. Thus $U_1 \cup V = U_2$ would be an annulus of orbits about $\mathcal{O}(x)$, with $U_1 \subset U_2$, and $U_2 - \bar{U}_1$ a nonempty open set. Continue in this manner, transfinitely if necessary, until we have exhausted the possibility of continuing. We then have a system $\{U_\alpha\}$ with each U_α an open annulus of orbits about $\mathcal{O}(x)$ and $U_\alpha \subset U_{\alpha+1}$, all α , and each $U_{\alpha+1} - \bar{U}_\alpha$ a nonempty open set. By this last con-

⁽³⁾ This theorem is subject to a proof not involving the continuity of the flow. A conjecture by R. H. Bing proved by this author states that an open connected set in the plane which is the union of disjoint simple closed curves must be an annulus. The proof involves heavier topological machinery than the author wishes to employ here.

dition, there can be only countably many U_α . Count them as $\{U_i\}$ and discard those which are contained in earlier ones. Then $U_{ij} \subset U_{i+j+1}$ and $U = \bigcup_{j=1}^{\infty} U_{ij} = \bigcup_{i=1}^{\infty} U_i = \bigcup_{\alpha} U_{\alpha}$. Then U is an open annulus of orbits about $\mathcal{O}(x)$, and every boundary point of U , by the condition of exhaustion, is a fixed point. Thus U is the component of x in the set of nonfixed points. It is thus shown that the set of nonfixed points is a union of disjoint open annuli, as required. Q. E. D.

6. THEOREM. *Let G be a set which is the disjoint union of open annuli. Then there is a continuous flow ϕ in E^2 with the properties that*

- 1° every point of G' is fixed under ϕ ,
- 2° no point of G is fixed under ϕ ,
- 3° every orbit of ϕ is compact⁽⁴⁾.

Proof. This proof can be made under additional restrictive hypotheses, and we will require, as an example, that the orbits of the flow be the images under analytic mappings of circles in the plane. Now it is sufficient to show that for any topological annulus, G_0 , it is possible to define a continuous flow on \bar{G}_0 such that the above conditions on the orbits are satisfied and such that the fixed points are precisely the points of $\bar{G}_0 - G_0$. But this is easily done, for the conformal mapping theorem assures us that we can map G_0 conformally onto the interior of a genuine annulus (the open set between two concentric circles, where 0 and ∞ are included as possible radii). For convenience, we will take these circles as being centered at the origin. Let g be the mapping from G_0 onto this annulus, and let r_0 and r_1 be the radii of the bounding circles. Then for each $r_0 < r < r_1$, the circle $|z| = r$ is the image of a simple closed curve which satisfies the augmented conditions of this theorem. G_0 is the union of these curves, and we will define a flow ϕ for which these are the orbits. Each of the curves is compact, so that it has a positive distance from G'_0 . If we define $f(x)$ = distance between the curve containing x and G'_0 , then f is continuous and positive in G_0 . Each of these circles in G_0 is now rectifiable, and so we can define $\phi(t, x)$ for small t at least by requiring that the arc length from x to $\phi(t, x)$ in the positive direction be equal to $tf(x)$. ϕ is then clearly a continuous flow in G_0 and if $x \in \bar{G}_0 - G_0$ and $x_i \rightarrow x$, with $x_i \in G_0$, and if we have a bounded sequence $\{t_i\}$ of real numbers, then it is clear that $f(x_i) \rightarrow 0$, so that $d(x_i, \phi(t_i, x_i)) \rightarrow 0$, and $\phi(t_i, x_i) \rightarrow x$. This tells us that the flow ϕ is continuous to fixed points in the boundary of G_0 .

If we are not interested in having such smooth orbits, then we need not employ such heavy machinery. Let h be a homeomorphism of G_0 onto a genuine annulus, say $1 < |z| < 2$. If we define

$$\phi_1(t, z) = h^{-1}(e^{it}h(z)),$$

⁽⁴⁾ Corrections in §§5, 6, 10, 11 and 15 were added in proof, November 23, 1964.

then ϕ_1 is a continuous flow in G_0 , but in general not extendable to fixed points on the boundary of G_0 . However, if we have any continuous function g from the reals into the reals, then

$$\phi_g(t, z) = h^{-1}(e^{itg(|h(z)|)}h(z))$$

is also a continuous flow in G_0 , and if g is properly chosen, as we shall see, then $\phi_g = \phi$ is continuous to the boundary.

For each integer $n \geq 3$, let D_n be the set $1 + 1/n \leq |z| \leq 2 - 1/n$. Then D_n is compact. For each $z \in D_n$ there is an $\epsilon_z > 0$ such that

$$0 \leq t < \epsilon_z \implies d(h^{-1}(z), h^{-1}(e^{it}z)) < d(h^{-1}(D_n), G'_0).$$

From the compactness of D_n , we conclude that it is possible to replace all the ϵ_z with a single $\epsilon_n > 0$.

Let $g_1(z) = \sup(\epsilon_n |z \in D_n)$. Clearly, $g_1(z) = g_1(|z|)$ is a step function which increases as $|z|$ increases from 1 to $3/2$ and decreases as $|z|$ increases from $3/2$ to 2. Let $g(z) = g_1(|z|)$ be any continuous function for which $0 < g(|z|) < g_1(|z|)$ on all of G_0 . We assert that this is the desired function. Since g is continuous, this will clearly give a continuous flow in G_0 . It is only necessary to check the boundary of G_0 . Let $x \in G'_0$ and let $x_j \rightarrow x$ with $x_j \in G_0, j = 1, 2, \dots$.

Let $t_j \rightarrow t \in R$. Then we shall show

$$\phi(t_j, x_j) = h^{-1}(h(x_j) \exp(it_j g(|h(x_j)|))) \rightarrow x.$$

Let N be an integer chosen so that $|t_j| < N, j = 1, 2, \dots$.

We note that if $z \in D_n - D_{n-1}$, then

$$d(h^{-1}(z), h^{-1}(ze^{itg(z)})) \leq d(h^{-1}(D_n), G'_0) \text{ if } |t| < 1.$$

Thus $d(h^{-1}(z), h^{-1}(ze^{itg(z)})) \leq N(d(h^{-1}(z), G'_0))$ if $|t| < N$. Therefore

$$d(x_j, \phi(t_j, x_j)) = d(x_j, h^{-1}(h(x_j) \exp(it_j g(h(x_j))))) < Nd(x_j, x),$$

and $\phi(t_j, x_j) \rightarrow x$, which completes the proof. Q. E. D.

It is also possible to accomplish this same result with the orbits being polygons, in a manner we will demonstrate later.

Let us now examine the set F of points characterized by Theorem 5. It is closed, so that all the components are closed. Let the union of the unbounded components be denoted as F_∞ . If F_∞ is not empty, then adding the point at ∞ to the plane (and to F), F_∞ is the component of ∞ in F . Thus F_∞ is closed in the plane. If we remove F_∞ from the plane, we are then left with an open, simply-connected set, which is now a finite or infinite union of pairwise disjoint open 2-cells. In each of these open cells, F is the complement of a disjoint union of annuli, and each component of F is compact. This leads us to

7. DEFINITION. If C is an open 2-cell and $F \subset C$ and $C - F$ is a dis-

joint union of open annuli, and if every component of F is compact, then F is called a T -set of C .

This definition gives us immediately

8. **THEOREM.** Let $\{C_i\}$ be a (finite or infinite) sequence of pairwise disjoint open 2-cells. Let $F_i \subset C_i$ be a T -set of C_i , $i = 1, 2, \dots$. Let $F_\infty = (\bigcup_i C_i)'$. Then $F = F_\infty \cup \bigcup_i F_i$ is the invariant set of a flow with compact orbits, and every such invariant set can be so represented.

Proof. Immediate from Theorems 5 and 6 and Definition 7.

We now turn our attention to the flows with closed orbits. In this case, the orbits can be points or circles or lines, but the lines must have their endpoints at ∞ . The key lemma is:

9. **LEMMA.** Let ϕ be a flow with closed orbits and let F be the invariant set of ϕ . Let F_∞ be the union of the unbounded components of F . Let $\{x_i\}$ be a sequence of elements of E^2 with $\mathcal{O}(x_i)$ not compact (i.e., a line), $i = 1, 2, \dots$. If $\lim_i x_i = x \in F$, then $x \in F_\infty$.

Proof. Assume not. Let F_0 be the component of x in F and the denial of this lemma would require F_0 to be compact. Let C_0 be a circle containing F_0 in its interior, together with all the x_i . Let $t_i < 0$ be chosen so that $\phi(t_i, x_i) \in C_0$, while $\phi(t, x_i) \notin C_0$ when $t_i < t \leq 0$. Then some subsequence of the sequence $\{\phi(t_i, x_i)\}$ converges to a point of C_0 , by the compactness of C_0 , and it will not disturb the generality of this proof to assume that the sequence itself converges, say to $x_0 \in C_0$. Look at all the sequences $\{\phi(s_i, x_i)\}$ where for each i , $t_i \leq s_i \leq 0$. Let F_1 be the set of all limits of such sequences. F_1 is clearly closed, and also connected. Furthermore, $x \in F_1$ and $x_0 \in F_1$. If $F_1 \subset F$, then $F_1 \subset F_0$ and $x_0 \in F_1 \cap C_0 \subset F_0 \cap C_0$ which is empty by the choice of C_0 , giving us a contradiction. Thus, there is a sequence $\{\bar{t}_i\}$ with $t_i \leq \bar{t}_i \leq 0$ such that $\bar{y} = \lim_i (\phi(\bar{t}_i, x_i)) \notin F$. It is clear from the invariance of x that $\bar{t}_i \rightarrow -\infty$ as $i \rightarrow \infty$, for otherwise \bar{y} could be obtained as $\phi(t, x)$ for some t . We now observe that, for every $t > 0$, $\bar{t}_i \leq t + \bar{t}_i \leq 0$ for all but finitely many values of i , and thus $\phi(t, \bar{y}) \in C_0 \cup \text{Int}(C_0)$ for all $t > 0$. Thus, the orbit of \bar{y} is not a line with end points at ∞ , and by Lemma 4 is not a circle. Thus, it is a single point, but this contradicts the choice of $\bar{y} \notin F$. This contradiction gives us the lemma. Q. E. D.

Let G_1 be the set of all points whose orbits are not compact. Then every fixed point in \bar{G}_1 is in F_∞ , while every nonfixed point in \bar{G}_1 is in G_1 , by Lemma 4. This tells us that $G_1 \cup F_\infty$ is closed in E^2 . We clearly see that no component of $G_1 \cup F_\infty$ is bounded, so that $E^2 - (G_1 \cup F_\infty)$ is the union of pairwise disjoint open 2-cells. In each of these open cells, the flow has only compact orbits and it is clear that F in each such cell has only compact components. Thus we see that F in each of these open cells is a T -set, and that these T -sets, together with F_∞ , make up all of F .

If we consider $E^2 - F_\infty$, this is a disjoint union of open cells C_i . The further removal of G_1 leaves in each C_i a disjoint union $\bigcup_j C_{i,j}$ of open cells. This may be an empty set, or a finite or infinite union. In any case, $F \cap C_i = \bigcup_j F \cap C_{i,j}$ and $F \cap C_i$ is closed in C_i . We adopt a name for such sets:

10. DEFINITION. Let C be an open 2-cell and let $\{C_j\}$ be a sequence (empty, finite, or infinite) of pairwise disjoint open 2-cells contained in C . For each j , let F_j be a T -set of C_j . Then if $F = \bigcup_j F_j$ is closed in C , we say that F is a T_σ -set of C . If C contains an arc which has an endpoint at ∞ , then C is said to be *arcwise connected to ∞* . If C is arcwise connected to ∞ and $F \subset C$, and if $C - F$ is not arcwise connected to ∞ , then F *shields ∞ in C* .

Using the definition, we prove

11. THEOREM. *If ϕ is a flow with closed orbits and F is the invariant set of ϕ , let F_∞ be the union of the unbounded components of F . Then in every component of $E^2 - F_\infty$ which is not arcwise connected to ∞ , F is a T -set, while in every component which is arcwise connected to ∞ , F is either a T -set or a T_σ -set which does not shield ∞ .*

Proof. The discussion above shows that in each component of $E^2 - F_\infty$, F is a T_σ -set. However, if C_i is not arcwise connected to ∞ , the removal of G_1 leaves it unchanged, so that F is a T -set of C . In case C is arcwise connected to ∞ , then either $C_i \cap G_1$ is empty or not. If it is, then F is again a T -set of C_i . If not, then F , which is a T_σ -set of C_i , does not shield ∞ , since every orbit contained in $C_i \cap G_1$ contains an arc with an endpoint at ∞ which does not intersect F . Q. E. D.

We now turn to the converse. Let $\{C_i\}$ be a sequence of pairwise disjoint open 2-cells. Let $F_i \subset C_i$ be chosen so that if C_i is not arcwise connected to ∞ , then F_i is either a T -set of C_i or a T_σ -set which does not shield ∞ , and if C_i is not arcwise connected to ∞ , then F_i is a T -set. We already know that in case F_i is a T -set, we can define in C_i a flow which will have compact orbits and which will have F_i for its invariant set, and will be continuous to fixed points on the boundary. We now turn to the remaining case, where F_i is a T_σ -set of C_i which does not shield ∞ , and show the analogous result there.

First, however, we will need some information on the structure of T_σ -sets.

12. LEMMA. *If C is an open 2-cell, and F is a T -set of C , and if F is not compact, then every component of $C - F$ has compact closure in C .*

Proof. $C - F$ is the union of disjoint open annuli. Assume the lemma is false, i.e., F is not compact and U_1 , a component of $C - F$, does not have a compact closure in C . For convenience, we can take C to be the open unit disc; this does not affect the generality of the result. The boundary of U_1 in the whole plane is made up of two connected pieces, each of them com-

pact. Since \bar{U}_1 is compact in the plane, but not in C , there must be a point of \bar{U}_1 on the unit circle K . Let K_1 be any simple closed curve in U_1 whose interior is not wholly contained in U_1 . Then K_1 contains in its interior one connected piece of the boundary of U_1 . Also, since F is not compact, there is a point $x \in F$ which lies outside K_1 . Let $y \in K_1$ be chosen so that $d(x, y) = d(x, K_1)$ and let us examine the line segment from x to y . It is clear that this segment contains a point of the boundary of U_1 , call it \bar{y} , and that the component B_1 of \bar{y} in the boundary of U_1 in the plane contains a point of K . It follows at once that the component of \bar{y} in $B_1 - K$ is not closed in the plane and thus not compact. Therefore, the component of \bar{y} in F is not compact, contrary to the assertion that F is a T -set. Q. E. D.

13. LEMMA. Let F be a T_σ -set of C : let $F = \bigcup_i F_i$, $C \supset \bigcup_i C_i$ (disjoint union), with F_i a T -set of C_i , all i . If $C_1 \neq C$, then F_1 is compact.

Proof. Assume that the lemma fails. Then since $C_1 \neq C$, C_1 has a boundary point in C , call it b . Let V be any (circular) neighborhood of b . Then V contains a point y of C_1 . Either $y \in F_1$ or $y \in C_1 - F_1$. If $y \in C_1 - F_1$, then the component U_j of y in $C_1 - F_1$ has compact closure in C . Let b_1 be the boundary point of C_1 which is closest to y on the line segment from b to y . Then the segment from b_1 to y contains a boundary point of U_j , which is a point of F_1 . Thus we see that V always contains a point of F_1 , and since F is closed, $b \in F$. However, since F is a T_σ -set and $b \notin C_1$, b must lie in a disc disjoint from C_1 , which is impossible. This proves the lemma. Q. E. D.

14. COROLLARY. Let F be a T_σ -set of C . Let $F = \bigcup_i F_i$, as in the above lemma. If F is not a T -set of C , then each F_i is compact.

Proof. For any j , a proper renumbering of the F_i , together with Lemma 13, will give the result.

15. THEOREM. Let $\{C_i\}$ be a (finite or infinite) sequence of disjoint open 2-cells in the plane, and let $F_\infty = (\bigcup_i C_i)'$. In case C_i is arcwise connected to ∞ , let $F_i \subset C_i$ be either a T -set of C_i or a T_σ -set which does not shield ∞ ; otherwise, let $F_i \subset C_i$ be a T -set of C_i . Then the set

$$F = F_\infty \cup \bigcup_i F_i$$

is the invariant set of a flow with closed orbits, and every such invariant set is of this form.

Proof. The second part of the conclusion has already been shown (Theorem 11). We will now demonstrate the first part. We will do so under the added restriction that the orbits must be polygonal paths. In order to do so, we will need to establish the following facts about polygonal flows. Let $\dots, p_{-2}, p_{-1}, p_0, p_1, p_2, \dots$ be a broken line made of infinitely many segments which does not intersect itself and runs from ∞ to ∞ . Let

$\dots, q_{-2}, q_{-1}, q_0, q_1, q_2, \dots$ be another disjoint from the first. Then the region between them can be mapped by a piecewise linear homeomorphism ψ_1 onto the strip $\{(x, y) | 0 \leq y \leq 1\}$ in the plane in such a way that each of the vertices occurs in the triangulation, and none of the dividing lines between triangles in the image is horizontal. Let the triangles of the image triangulation be counted as $\Delta_1, \Delta_2, \dots$, and let v_1, v_2, \dots be any sequence of positive numbers. We can define a flow ϕ_1 in the image strip by having each point in Δ_k move from left to right with velocity v_k for all k . Then $\phi(t, x) = \psi_1^{-1}(\phi_1(t, \psi_1(x)))$ defines a flow in the region between the broken lines, and a proper choice of the v_k will give us any desired velocities on the segments $[p_i, p_{i+1}]$, $[q_j, q_{j+1}]$ and we can easily arrange that the sup and inf of the velocities in the region are the same as those for the broken lines. We see that each orbit under ϕ is a polygonal path.

Let P_1 be a polygon and let P_2 be a polygon contained in the interior of P_1 . Let Q_1 be a square and Q_2 another square concentric with Q_1 and with parallel sides contained in the interior of Q_1 . Let ψ_2 be a piecewise linear homeomorphism of the annulus between P_1 and P_2 onto the annulus between Q_1 and Q_2 such that every vertex of P_1 and P_2 is contained in the triangulation, so that none of the sides between triangles in the image is horizontal or vertical, and so that the diagonal segments between the squares are included (piecewise) in the triangulation of the image. Let $\Delta_1 \dots \Delta_n$ be the triangles of the image, and let v_1, \dots, v_n be any positive numbers. The diagonals divide the annulus between Q_1 and Q_2 into four trapezoidal regions. We now have each point in Δ_k move with speed v_k , for all k , the points in the top trapezoid moving right to left, the points in the bottom trapezoid left to right, left trapezoid top to bottom, right trapezoid bottom to top. Then regardless of the choice of the v_k 's, this defines a continuous flow ϕ_2 with compact orbits. Let $\phi(t, x) = \psi_2^{-1}(\phi_2(t, \psi_2(x)))$. Then if the v_k 's are properly chosen, we can obtain any desired speeds on the sides of P_1 and P_2 , and also assure that the maximum and minimum speeds for the annulus are attained on the boundary. It is clear that each orbit under ϕ is a polygon (of no more than n sides). Note that in this construction, any pair of rectangles, one within the other with parallel sides, would have done equally well instead of the squares, *mutatis mutandis*. Let $(\dots, p_{-1}, p_0, p_1, \dots)$ and $(\dots, q_{-1}, q_0, q_1, \dots)$ be nonintersecting simple broken lines, with ends at ∞ , as before, and let P be a polygon lying in the open region between them. Let ψ_3 be a piecewise linear homeomorphism of the annulus thus formed onto the strip $\{(x, y) | -1 \leq y \leq 2\}$ exclusive of the open unit square $Q = (0, 1) \times (0, 1)$. Let us consider the rectangles concentric with and parallel to Q having length n and width $3 - 2/n$ for each $n = 1, 2, \dots$. Then we can choose our mapping ψ_3 so that each of the "diagonal" segments joining corresponding corners of adjacent rectangles is included piecewise in the image triangu-

lation, and further, that, as before, no side between triangles in the image is horizontal or vertical, and that each vertex of each broken line and of P is included in the triangulation. Let the triangles of the image triangulation be $\Delta_1, \Delta_2, \dots$ and let v_1, v_2, \dots be any positive numbers. Then any point x lies in the top, bottom, right, or left part of the annulus between adjacent rectangles defined above, and we define a flow ϕ_3 by requiring that x move respectively to the left, right, up or down with speed v_k if it lies in triangle Δ_k , for all k . Then again, $\phi(t, x) = \psi_3^{-1}(\phi_3(t, \psi_3(x)))$ is a continuous flow in the original space and the v_k can be chosen to give any desired velocities on the pieces of the boundary, and the speeds on the boundary can be the extremes for all speeds in the region.

Note that in the three examples given, not only are all the orbits polygonal, but the mapping $X \times R \rightarrow X$ is itself piecewise linear. Such a flow will be called a *regular polygonal flow*. Let A be any annulus. Then we can fill A with polygonal annuli $\dots, A_{-1}, A_0, A_1, \dots$ and define in each annulus a regular polygonal flow so that the speed of flow on annulus A_k does not exceed $1/|k|$ and is exactly $1/|k|$ on the boundary closest to A_0 and exactly $1/(|k| + 1)$ on the other boundary. Then all of these flows taken together define a regular polygonal flow in A which is continuous to fixed points on the boundary.

In like manner, suppose that G is an open 2-cell which is arcwise connected to ∞ and F is a T_σ -set of G . Let $\{C_i\}$ be a set of open cells such that each F_i is a T -set of C_i and $\bigcup_i F_i = F$. If F is a T -set of G , then the previous observation allows us to establish a regular polygonal flow in G with compact orbits having F and the boundary of G as fixed points. If F is not a T -set of G , then F is a T_σ -set which does not shield ∞ and we designate $F = \bigcup_i F_i$ with each F_i a T -set of C_i , and the C_i disjoint. Since F is not a T -set of G , each F_i can be enclosed in a polygon Q_i contained in C_i . Since G is σ -compact, let $\{K_j\}$ be an expanding sequence of compact sets whose union is G . We can then take Q_i so that Q_i is disjoint from K_j whenever F_i is. We now choose any simple broken line P_0 which runs from ∞ to ∞ in G and does not intersect any of the Q_i . Let P_1 be a similar broken line disjoint from P_0 and not intersecting any of the Q_i so chosen that of all the Q_i , only Q_1 lies between P_0 and P_1 (unless F is empty, in which case this last requirement is dropped). Let P_2 be chosen so that it lies in the region on the other side of P_1 from P_0 and does not intersect P_1 and so that, if F is not empty on that side of P_1 , the region between P_1 and P_2 contains exactly one of the Q_i . Likewise P_{-1} is chosen on the side of P_0 away from P_1 and contains exactly one of the Q_i , between P_{-1} and P_0 , unless F is empty in the indicated region. We continue in this manner, obtaining a P_i for each $-\infty < i < +\infty$. It is easily seen that any compact set $K \subset G$ can intersect only finitely many of the F_i since other-

wise the sets $(\bigcup_{i>j} F_i) \cap K$ would have the finite intersection property and thus a limit point. This limit point would have to lie in $(\bigcup_i C_i)'$ and not be in $\bigcup_i F_i$, contrary to the assertion that $\bigcup_i F_i$ is closed. Thus, any one of the K_j meets at most finitely many F_i , and thus finitely many Q_i , and we can thus arrange that each K_j meets only finitely many P_i , and this will then automatically be true for any compact set $K \subset G$. Thus, each of the Q_i lies between two adjacent broken lines, and we renumber the Q_i so that Q_i lies between P_{i-1} and P_i , renumbering the F_i and C_i consistently with the Q_i . Define a sense on P_0 and let a regular polygonal flow be defined on the region between P_0 and P_{-1} , and excluding the interior of Q_0 (if there is a Q_0) which has speed 1 on all orbits. The sense of the flow on P_{-1} will be determined by whether there is a Q_0 . In the annulus between Q_0 and F_0 , we can, by using methods described above, define a regular polygonal flow which is continuous to fixed points on the F_0 boundary and has speed 1 on Q_0 and such that the sense agrees with that already defined on Q_0 . We can then define a regular polygonal flow in the rest of C_0 giving F_0 as the invariant set. If there is no Q_0 , define a regular polygonal flow in the strip between P_{-1} and P_0 which has speed 1 on all orbits. Continuing, we obtain a flow on the (possibly punctured) strip between P_{-2} and P_{-1} which agrees with the previously defined flow on P_{-1} and has speed $1/2$ on P_{-2} . If there is a Q_{-1} , the speed along Q_{-1} is 1, and we fill in the interior of Q_1 similarly, with 1 an upper bound for the speed of all orbits lying between P_{-2} and P_{-1} . We proceed sequentially for all the strips between P_{i-1} and P_i , $i < 0$, and the speed along P_i and Q_i is always $1/|i|$, and the upper bound for speeds in the strip is always $1/|i|$. We do the same for the strips between P_{i-1} and P_i , $i > 1$, with increasing i , except that we bound the speeds in that strip by $1/(i-1)$. It is easily seen that this is a regular polygonal flow, continuous to fixed points on the boundary of G , and that in each strip the fixed points, if any, are one of the F_i . All the F_i are accounted for, and we have thus shown that for any open cell G and any set $F \subset G$ which meets the hypotheses of the theorem, according as G is or is not arcwise connected to ∞ , we can produce a regular polygonal flow in G which has F for its fixed-point set and all its orbits closed (resp. compact), and is continuous to fixed points on the boundary of G . Furthermore, the maximum speed of this flow is 1, and by a small change we could make it any desired number.

Returning to the notation of the statement of the theorem, if F_i is a T_σ -set (resp. T -set) of C_i , then there is a regular polygonal flow in C_i with closed orbits having maximum speed $1/i$ and having F_i for its fixed-point set. If we construct a flow by combining these and making all the points of $F_\infty = (\bigcup_i C_i)'$ fixed points, then it is easy to see that the resulting flow has only closed orbits and has F for its fixed-point set. The changes necessary to have analytic curves instead of polygonal lines are obvious

extensions of methods used earlier. Q. E. D.

We note that at least one of the boundaries of any annulus must have uncountably many points, so that a countable T -set must consist of exactly one point. Thus:

16. COROLLARY. *If F is a countable set which is an invariant set of a flow with closed orbits, F is a discrete set.*

Proof. Each point is an F_i and since $F_i \subset C_i$, is open in F . Q. E. D.

17. COROLLARY. *A convergent sequence of infinitely many distinct points together with the limit is not the invariant set of flow with closed orbits.*

Proof. Immediate consequence of Corollary 16.

18. **Extensions of this work.** It is natural to ask what are the extensions of these results, and these have proceeded in two directions. The author has continued to work in the plane, and has obtained for flows with no stagnation points (a stagnation point is a fixed point which is a limit point of a line orbit) results similar to those reported in this paper. Further, flows with finitely many stagnation points, or a discrete set of stagnation points, have been studied and the results are, again, similar. It is known that this does not extend as far as flows with countably many stagnation points, and in fact, for any closed set F in the plane, there is a flow ϕ such that ϕ has only countably many stagnation points and F is the invariant set of ϕ . Between the case of discrete stagnation sets and countable ones, there is some ground to be covered, but it now appears that each case will either show the character of the compact orbit case or else allow all closed sets as invariant sets. The author has a paper in preparation on this subject [3].

Dr. Ta-Sun Wu, in his dissertation written at Tulane University, extends the results of this paper to 2-manifolds in general, by use of heavier topological machinery than is employed here. The rôle of annuli in this paper is taken, in his work, by either annuli or Möbius bands, and he notes, after suitable lemmas have been established, that the same analysis as this author employs will prove an analogous result (cf. [4]).

The case of flows in 3-space is immensely important. It is clear that the methods used herein are essentially 2-dimensional and do not generalize readily. Any results in 3-space would be very welcome indeed.

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UNIVERSITY OF WISCONSIN,
MADISON, WISCONSIN